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Use of AL-KARAJI method in calculating sum

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Abstract: This article provides several ways to find the sum of some number series. Al-Karaji's contribution to the science of mathematics lies in the method that he used to calculate the sums, and finding of the sum of a series of cubes of natural numbers is proved in this way. To explain al-Karaji's method more widely, other sums were calculated in the same way, and a general result was obtained.

Keywords: Number series, Newton binomial, binomial coefficient, arithmetic progression, short multiplication formula, rectangle, square.

Introduction: The development of Arab mathematics began in the 7th century AD, just in the era of the emergence of the religion of Islam. It grew out of the many challenges posed by trade, architecture, astronomy, geography, optics, and deeply combined the desire to solve these practical problems and intense theoretical work. Arab mathematicians achieved significant achievements and made a number of undeniable discoveries in the development of algebraic calculus, both abstract and practical, the formation of the theory of equations, algorithmic methods at the junction of algebra and arithmetic. In the development of Arab mathematics, two stages can be distinguished: first of all, the assimilation in the 7th and 8th centuries of the Greek and Eastern heritage. Baghdad was the first major scientific center during the reign of al-Mansur (754-775) and Harun al-Rashid (786-809). There were a large number of libraries, and many copies of scientific works were made. The works of ancient Greece (Euclid, Archimedes, Apollonius, Heron, Ptolemy, Diophantus) were translated, and works from India, Persia and Mesopotamia were also studied. But by the 9th century, a real Arab mathematical culture of its own had formed, and new work went beyond the limits defined by the Hellenic mathematical heritage. The first famous scientist of the Baghdad school was Muhammad al-Khwarizmi, whose activity took place in the first half of the 9th century. He was part of a group of mathematicians and astronomers who worked in the House of Wisdom, a kind of academy founded in Baghdad during the reign of al-Mammun (813-833). Five works by al-Khorezmi have survived, partially revised, of which two treatises on arithmetic and algebra had a decisive impact on the further development of mathematics. His treatise on arithmetic is known only in the Latin version of the 13th century, which, no doubt, is not an accurate translation. It could be titled "A Book on Addition and Subtraction Based on Indian Calculus." This is, in any case, the first book that sets out the decimal number system and the operations performed in this system, including multiplication and division. In particular, a small circle was used there, which served as a zero. Al-Khorezmi explained how to pronounce numbers using the concepts of one, ten, hundreds, thousands, thousands of thousands ... that he defined. But the form of the numbers used by al-Khorezmi is unknown, perhaps they were the letters of the Arabic alphabet or the Arabic numerals of the East. In fact, a purely alphabetic number system existed for a very long time, as evidenced by the "Book of arithmetic for scribes and traders", written by Abu-l-Wafa between 961 and 976, and the famous "A sufficient book on the science of arithmetic," written by al-Karadzhi at the end of the X - beginning of the XI century. **[1-3]**

Al-Karaji (late X - early XI centuries), a native of the city of Karaj, located between Tehran and Qazvin, is the author of many very important works, namely, "A Sufficient Book on the Science of Arithmetic", "Al-Fakhri", an extensive algebraic treatise, dedicated to the vizier of Baghdad Fakhr al-Mulk, as well as the book "Al-Badi" devoted to the study of indefinite equations. The Sufficient Book is a textbook of practical arithmetic, much like another book he wrote between 961 and 976. Abu-l-Wafa and which bears the name "Book of arithmetic for scribes and merchants". The numbers were written there verbally and nowhere was the decimal positional number system used, which was more in line with the habits of traders. Abu-l-Wafa considered the theory of fractions in detail. Al-Karaji also paid attention to the decomposition of ordinary fractions into the sum of aliquot fractions. Note here that at the end of the 10th century, arithmetic algorithms, in particular, algorithms for extracting square and even cube roots, were significantly developed. Al-Uqlidizi (about 952-953) gave an approximation with a lack of square root of the expression $\sqrt{N} = a + r/(2a+1)$. Other mathematicians, such as Kushiyar ibn Labban and his student An-Nasawi, improved these results and extended them to the cube root using the decimal representation of the number $N = n_0 10^{m-1} + ... + n_m$ and the decomposition of the binomial $(a+b)^3$ as well as $(a+b+...k)^3$.**[3]**

The successful development of arithmetic algorithms led Al-Karaji and his followers to search for similar procedures in the case of algebraic expressions. In addition to the practical part, the "Sufficient Book" contains the main algebraic part devoted to solving six canonical types of equations. But his presentation is an achievement from the point of view of methodology, because al-Karaji grouped before each problem the elements of algebraic calculus that are essential for its solution (transformation of irrational quantities, identities, etc.). This theoretical orientation was clearly established in the algebraic treatise "Al-Fakhri". In his preface, al-Karaji defined the goal of the science of calculus as determining indefinite quantities using known ones. It is necessary to turn to the means of arithmetic calculus and apply them to all expressions containing unknowns. Thus algebra became explicitly the arithmetic of the unknown. It can be said that here its subject was defined for the first time, and the al-Karaji school expanded the range of methods and algorithms applied to expressions containing the unknown. **[4-8]**

He applied arithmetic operations to monomials, and then to expressions composed of monomials, that is, to polynomials, considering addition and subtraction on equal terms. As for division, he limited himself to division into monomials. We get acquainted with the results of the al-Karaji school concerning division and square root extraction from the work of his follower al-Samawal, who continued his research.

But al-Karaji has already managed to describe what could now be called the algebra of polynomials. These methods of "arithmetization of algebra", according to Rashed, are based, on the one hand, on the initial elements of the algebra of al-Khorezmi and Abu-Kamil, and on the other hand, on the translation of Diophantus, performed by Costa ibn Luke under the title "The Art of Algebra". Indeed, although "Arithmetic" considered arithmetic on the set of positive rational numbers, Diophantus used methods of an algebraic nature in it. These methods influenced the methods of the Arab algebraists of the second period, who mastered and developed them. Al-Karaji summed up many finite arithmetic series, such as for which he gave a beautiful proof, both geometric and algebraic. In the text of al-Samawal, which he, however, attributed to al-Karaj, there is a table of coefficients for $(a+b)^n$ to $n=12$, and the author adds that it can be continued indefinitely in accordance with the rule of formation, which is now written as $C_n^m = C_{(n-1)}^{(m-1)} + C_{(n-1)}^m$ (the so-called Pascal's triangle).

Karaji defined algebra as "a method of calculus that allows, using known quantities, to find unknowns." It should be considered his merit that he began to solve algebraic problems exclusively by mathematical methods, without resorting to geometric ones. Thus, Karaji is the founder of the algebraic method without the use of geometric schemes. Thus he showed that algebra itself is a self-sufficient discipline.

Historian, orientalist and mathematician Franz Wöpke wrote about al-Karaji: "This is the first mathematician who proposed the most advanced algebraic theories of calculus in the Islamic world." Al-Karaji created a table of binomial coefficients, the principle of their additive generation and a binomial formula. Al-Karaji was the first to systematically use algebraic methods of calculus, worked with definite and indefinite equations, derived not only square, but also cube roots. **[9-11]**

He took algebra beyond the bounds of Euclidean geometry and contributed to its formation as a separate discipline. Also, thanks to his work, algebra began to take the form it has now. In his "Book of Algebra and Muqabal" the scientist leads to determine the sum of an arithmetic progression, as well as the sum of squares and cubes of consecutive numbers.

Al-Karaji was not only a mathematician, but also a hydrologist. In his book On Finding Hidden Waters, he describes the physical characteristics and vegetation of the soils under which the water sources are located. He also gives definitions of water sources and gives groundwater extraction techniques. All this allows us to call him the first hydrological engineer.

METHODOLOGY

Al-Karaji systematically studied the algebra of indicators, and was the first to understand that the $x,\ x^2,\ x^3,\ldots$ sequence could be extended indefinitely; and backward $\frac{1}{x},\ \frac{1}{x^2},\ \frac{1}{x^3},\ldots$. However, since, for example, the product of a square and a cube will be expressed in words, and not in numbers, like a square-cube, the numerical property of adding indicators was not clear.

Al-Karaji gave the first formulation of binomial coefficients and the first description of Pascal's triangle. He is also credited with discovering the binomial theorem. Another important idea introduced by al-Karaji and continued by al-Samaw'al and another was that of an inductive argument for solving certain arithmetic sequences. Thus, al-Karaji used such an argument to prove a result about the sums of integral cubes already known. Al-Karaji did not, however, state the general result for an arbitrary n. He stated his theorem for a specific integer $n=10$. His proof, however, was clearly intended to be extensible to any other integer. Al-Karaj's argument includes essentially two main components of the modern argument by induction, namely the truth of the statement for $n=1\left(1=1^3\right)$ and the originating truths for $n=k$ from which of $n=k-1$. Of course, this second component is not explicit, since, in a sense, al-Karaj's argument is in the opposite direction; this, it starts at $n=10$ and goes down to $n=1$, not going up. However, his argument in al-Fakhri is the earliest surviving proof of the addition formula for integral cubes. **[12-14]**

Al-Karaji operates not only with square but also cube roots, using the formula for the cube of the sum and difference. He gives rules for determining the sum of an arithmetic progression, as well as the sum of squares and cubes of consecutive numbers. For the sum of squares, al-Karaji gives the correct formula, but says that he cannot prove it correct. For the sum of cubes, he gives a geometric proof. Al-Karaji gives in his essay a table of binomial coefficients, the principle of their additive generation and the binomial formula. **[15-21]**

Statement of Problem

How are the sums calculated?

If the word is about the sum of two or three people, it is understandable. But sometimes there are problems, such as calculating the sum of the terms of a sequence of large numbers with some connection. This is no longer an easy task. The problem of calculating a number of complex sums from the history of mathematics to the present has attracted the attention of many mathematicians. Examples include the manuscripts and historical problems that have come down to us. Here is one such issue.

Question: If the clock rings 1 time at 1, 2 times at 2, 3 times at 3, and so on, how many times a day will it ring?

In order to find the number of bells, just find the sum of the numbers from 1 to 12 and multiply by 2. These numbers are part of the arithmetic progression, so it's easier to find the sum. If we change the problem as follows, that is, if the clock rings 1 time when it is 1, 4 times when it is 2, in short, what is the square of the clock, how many times does it ring in a day? To solve this problem

$$\sum_{k=1}^{12} k^2 = 1^2 + 2^2 + 3^2 + \ldots + 12^2$$

You need to calculate the sum of the views and multiply it by 2. We can say that this is not easy anymore. The formulas for calculating such sums have been used since ancient times. In this article, we will focus on two methods of calculating such sums.

Using Newton's binomial formula for calculating sums

Example 1. Evaluate $S_n^1 = \sum_{k=1}^{n} k = 1+2+3+\ldots+n$.

We use Binomial theorem in order to solve the problem. Namely, From

$$(k+1)^2 = k^2 + 2\cdot k\cdot 1 + 1^2$$

We put $1,2,\ldots,n$ instead of k respectively and we rewrite following equalities

$$2^2 = 1^2 + 2\cdot 1\cdot 1 + 1^2,$$
$$3^2 = 2^2 + 2\cdot 2\cdot 1 + 1^2,$$

$$4^2 = 3^2 + 2 \cdot 3 \cdot 1 + 1^2,$$

$$\dots\dots\dots\dots\dots$$

$$(n+1)^2 = n^2 + 2 \cdot n \cdot 1 + 1^2.$$

We sum all equations and obtain following equalities.

$$2^2 + 3^2 + 4^2 + \ldots + (n+1)^2 = 1^2 + 2^2 + 3^2 + \ldots + n^2 + 2 \cdot (1 + 2 + \ldots + n) + n,$$

$$(n+1)^2 - 1^2 = 2 \cdot S_n^1 + n.$$

Since $(n+1)^2 = n^2 + 2 \cdot n + 1$ the last equality rewrites as follows

$$S_n^1 = \frac{(n+1)n}{2}. \tag{1}$$

Example 2. Evaluate $S_n^2 = \sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^2 + \ldots + n^2$.

In this example we use Binomial theorem in order to solve the problem. Namely,

$$(k+1)^3 = k^3 + 3 \cdot k^2 \cdot 1 + 3 \cdot k \cdot 1^2 + 1^3$$

We put $1, 2, \ldots, n$ instead of k respectively and we rewrite following equalities

$$2^3 = 1^3 + 3 \cdot 1^2 \cdot 1 + 3 \cdot 1 \cdot 1^2 + 1^3,$$

$$3^3 = 2^3 + 3 \cdot 2^2 \cdot 1 + 3 \cdot 2 \cdot 1^2 + 1^3,$$

$$4^3 = 3^3 + 3 \cdot 3^2 \cdot 1 + 3 \cdot 3 \cdot 1^2 + 1^3,$$

$$\dots\dots\dots\dots\dots\dots\dots$$

$$(n+1)^3 = n^3 + 3 \cdot n^2 \cdot 1 + 3 \cdot n \cdot 1^2 + 1^3.$$

We sum all equations and obtain following equalities.

$$2^3 + 3^3 + 4^3 + \ldots + (n+1)^3 = 1^3 + 2^3 + 3^3 + \ldots + n^3 + 3 \cdot \left(1^2 + 2^2 + \ldots + n^2\right) + 3 \cdot (1 + 2 + \ldots + n) + n,$$

$$(n+1)^3 - 1^3 = 3 \cdot S_n^2 + 3 \cdot S_n^1 + n. \tag{2}$$

From (1) we get $S_n^1 = \frac{(n+1)n}{2}$. We put $\frac{(n+1)n}{2}$ to (2) and find S_n^2

$$3 \cdot S_n^2 = (n+1)^3 - 1^3 - 3 \cdot \frac{(n+1)n}{2} - n,$$

$$3 \cdot S_n^2 = \frac{2n^3 + 3n^2 + n}{2},$$

$$S_n^2 = \frac{n(n+1)(2n+1)}{6}. \tag{3}$$

Hence, one gets $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$. Now, we can solve the above problem of clock by using formula (3). For $n = 12$ we get

$$\sum_{k=1}^{12} k^2 = 1^2 + 2^2 + 3^2 + \ldots + 12^2 = \frac{12(12+1)(2 \cdot 12 + 1)}{6} = 650$$

Since the number of all clock bells in 24 hours, obtained result is multiplied by 2. Hence, in all 1300 clock bells.

Example 3. Evaluate the sum $S_n^3 = \sum_{k=1}^{n} k^3 = 1^3 + 2^3 + 3^3 + \ldots + n^3$.

We use the same method as above. One uses the following equality

$$(k+1)^4 = k^4 + 4 \cdot k^3 \cdot 1 + 6 \cdot k^2 \cdot 1^2 + 4 \cdot k \cdot 1^3 + 1^4.$$

Namely,

$$2^4 = 1^4 + 4 \cdot 1^3 \cdot 1 + 6 \cdot 1^2 \cdot 1^2 + 4 \cdot 1 \cdot 1^3 + 1^4,$$

$$3^4 = 2^4 + 4 \cdot 2^3 \cdot 1 + 6 \cdot 2^2 \cdot 1^2 + 4 \cdot 2 \cdot 1^3 + 1^4,$$

$$4^4 = 3^4 + 4\cdot 3^3 \cdot 1 + 6\cdot 3^2 \cdot 1^2 + 4\cdot 3\cdot 1^3 + 1^4,$$

....

$$(n+1)^4 = n^4 + 4\cdot n^3 \cdot 1 + 6\cdot n^2 \cdot 1^2 + 4\cdot n\cdot 1^3 + 1^4.$$

We sum all equalities and obtain

$$(n+1)^4 - 1^4 = 4S_n^3 + 6S_n^2 + 4S_n^1 + n \tag{4}$$

We put (1) and (3) to (4) and find S_n^3.

$$4S_n^3 = (n+1)^4 - 1^4 - 6\frac{n(n+1)(2n+1)}{6} - 4\frac{(n+1)n}{2} - n,$$

$$4S_n^3 = n^4 + 2n^3 + n^2,$$

$$S_n^3 = \frac{n^2(n+1)^2}{4}. \tag{5}$$

Hence, $\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$.

Example 4. Evaluate $S_n^m = \sum_{k=1}^{n} k^m = 1^m + 2^m + 3^m + ... + n^m$.

For calculating the sum we use Binomial theorem, i.e.,

$$(a+b)^m = \sum_{k=0}^{m} C_m^k a^{m-k} b^k .$$

Here $C_m^k = \frac{m(m-1)\ldots(m-k+1)}{k!}$ binomial coefficient. Then we obtain

$$(1+1)^{m+1} - 1^{m+1} = C_{m+1}^1 \cdot 1^m \cdot 1 + C_{m+1}^2 \cdot 1^{m-1} \cdot 1^2 + ... + 1^{m+1}$$

$$(2+1)^{m+1} - 2^{m+1} = C_{m+1}^1 2^m \cdot 1 + C_{m+1}^2 2^{m-1} \cdot 1^2 + ... + 1^{m+1}$$

....

$$(n+1)^{m+1} - n^{m+1} = C_{m+1}^1 n^m \cdot 1 + C_{m+1}^2 n^{m-1} \cdot 1^2 + ... + 1^{m+1}$$

We sum all equalities and obtain.

$$(n+1)^{m+1} - 1^{m+1} = C_{m+1}^1 S_n^m + C_{m+1}^2 S_n^{m-1} + C_{m+1}^3 S_n^{m-2} ... + n \tag{6}$$

From (6) we find A_n^0. Clearly, if $k > m+1$, then $C_{m+1}^k = 0$. Also, if $m = 0$ then (6) can be written as $(n+1)^1 - 1^1 = C_1^1 S_n^0$. Consequently, we get $S_n^0 = n$. If we calculate of (6) for the case $m = 1$, then one gets

$$(n+1)^2 - 1^2 = C_2^1 S_n^1 + C_2^2 S_n^0$$

$$S_n^1 = \frac{n(n+1)}{2}$$

Namely, we prove first part of first example.

Now we put 2 instead of m in (6) and find S_n^2.

$$(n+1)^3 - 1^3 = C_3^1 S_n^2 + C_3^2 S_n^1 + C_3^3 S_n^0$$

By using the values of S_n^0 and S_n^1 we find the value of S_n^2.

$$S_n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Hence, corollary in the second problem is the same as (2).

By using the process, we can calculate S_n^m $(m \in N)$.

For instance, from

$$S_n^4=\sum_{k=1}^{n}k^4=1^4+2^4+3^4+...+n^4=\frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$$

We put necessary equations to (6).

Now we calculate the above summations with other methods. This method is called al-Karaji method or method of rectangle.

4. Al-Karaji method

At first, we begin calculation from simple summations.

Example 5. Calculate the sum $S_n^1=\sum_{k=1}^{n}k=1+2+3+...+n$.

In example 1, we have previously proved that this sum is equal to (1) using Newton's binomial formula. Now we're going to get the sum by using a rectangular way. Now we're going to get this sum in a rectangular way. To calculate this sum, we get a square of size $n\times n$ (Figure 1) consisting of n columns and rows. Obviously, inside the square, small squares are formed by the intersection of rows and columns. Then the total number of squares is equal to $n\cdot n=n^2$.

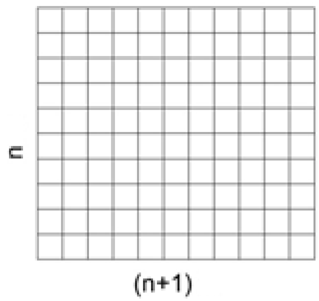


Figure 1

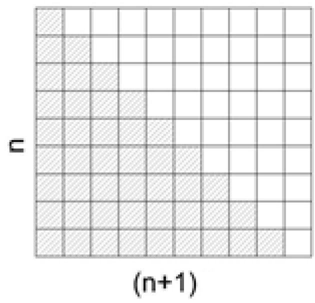


Figure 2

Now, we separate one row from the top of the square and one column from the right (Figure 2) and calculate the number of all squares inside the obtained domain.

$$n+n-1=2n-1$$

Hence, we have $2n-1$ squares in our first domain.

Similarly, we separate the same domain from other parts of the square (Figure 2) and calculate the number of squares in it.

$$(n-1)+(n-1)-1=2(n-1)-1$$

Continuing this process to the end, we write the obtained results. It should be noted that each time separated domains are called al-Karaji gnomons.

$$2n-1,\quad 2(n-1)-1,\quad ...,\quad 2\cdot 1-1$$

The sum of all these values is equal to the total number of squares in the square in Figure 1.

$$2n+2(n-1)+...+2\cdot 1-n=n^2$$

$$2(1+2+...+n)-n=n^2$$

$$1+2+...+n=\frac{n(n+1)}{2}$$

Hence, $S_n^1=\sum_{k=1}^{n}k=\frac{n(n+1)}{2}$.

Example 6. Calculate the sum $S_n^3=\sum_{k=1}^{n}k^3=1^3+2^3+3^3+...+n^3$.

This example was first considered by the Iranian mathematician Al-Karaji, using the rectangular method described above. This is why this method is called Al-Karaji method. For solving the above example, Al-Karaji took a square consisting of $(1+2+3+...+n)$ columns and rows on each side. (Figure 3)

The total number of squares is equal to

$$(1+2+3+...+n)(1+2+3+...+n)=(\frac{n(n+1)}{2})^2$$

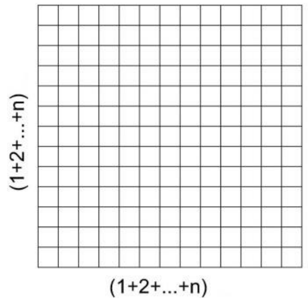


Figure 3

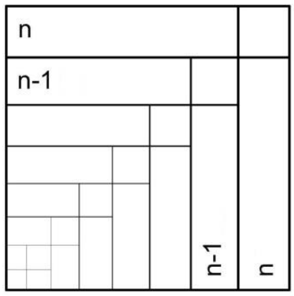


Figure 4

Now separate n rows and n columns and calculate the number of squares in it. (Figure 4)

$$2n(1+2+...+n)-n^2=2n\frac{n(n+1)}{2}-n^2=n^3+n^2-n^2=n^3$$

There are n^3 squares in our separated domain.

In the same way, we calculate the number of squares by separating $(n-1)$ rows and $(n-1)$ columns from the remaining domain.

$$2(n-1)(1+2+\ldots+(n-1))-(n-1)^2=2(n-1)\frac{n(n-1)}{2}-(n-1)^2=(n-1)^3$$

There are $(n-1)^3$ squares in this domain.

Continuing this process to the end and recording all the results. Then the following sequence of values is formed

$$n^3,\ (n-1)^3,\ (n-2)^3,\ldots,2^3,\ 1^3.$$

The sum of all these values is equal to the total number of squares in Figure 3.

$$1^3+2^3+3^3+...+n^3=(\frac{n(n+1)}{2})^2=\frac{n^2(n+1)^2}{4}$$

Hence, $S_n^3=\sum_{k=1}^{n}k^3=\frac{n^2(n+1)^2}{4}$ (see [3]) . We also solved this example with the short multiplication method above and got the same result (5) as the current one.

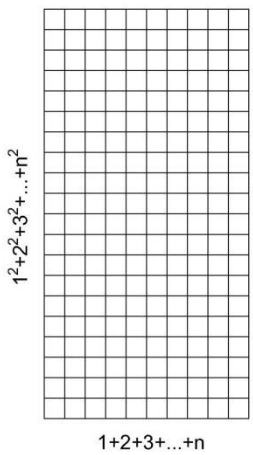
Al-Karaji calculated only S_n^3 with this method. Is it possible to calculate $S_n^4,S_n^5,S_n^6,\ldots,S_n^m,\ \ (m=1,2,3,4,\ldots)$ using this method as well? We have calculated above the cases $m=1$ and $m=3$.

Throughout the article, we show that all $S_n^4,S_n^5,S_n^6,\ldots,S_n^m,\ \ (m=2,3,4,\ldots)$'s can be calculated using the rectangular method. This is the novelty part of the article.

Example 7. Calculate the sum $S_n^4=\sum_{k=1}^{n}k^4=1^4+2^4+3^4+...+n^4$.

For this example, we get a rectangle with $(1^2+2^2+3^2+...+n^2)$ row of length and $(1+2+3+...+n)$ columns of width. (Figure 5) In a rectangle, a total of square is

$$(1^2+2^2+3^2+...+n^2)\cdot(1+2+3+...+n)=\frac{n(n+1)(2n+1)}{6}\cdot\frac{n(n+1)}{2}.$$



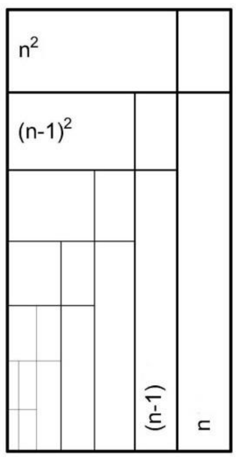


Figure 5 Figure 6

Now separate n columns from n^2 row and calculate the number of squares in it (Figure 6).

$$n^2\left(1+2+\ldots+n\right)+n\left(1^2+2^2+\ldots+n^2\right)-n^3=n^2\frac{n\left(n+1\right)}{2}+n\frac{n\left(n+1\right)\left(2n+1\right)}{6}-n^3=\frac{5}{6}n^4+\frac{1}{6}n^2$$

It turns out that there is $\frac{5}{6}n^4+\frac{1}{6}n^2$ squares in our area before separation. From the part of the rectangle outside the domain we have separated, we separate $(n-1)^2$ rows and $(n-1)$ columns and count the number of squares in it.

$$(n-1)^2\left(1+2+\ldots+(n-1)\right)+\left(n-1\right)\left(1^2+2^2+\ldots+\left(n-1\right)^2\right)-\left(n-1\right)^3=$$

$$=\left(n-1\right)^2\frac{n\left(n-1\right)}{2}+\left(n-1\right)\frac{n\left(n-1\right)\left(2n-1\right)}{6}-\left(n-1\right)^3=\frac{5}{6}(n-1)^4+\frac{1}{6}(n-1)^2$$

Hence, there is $\frac{5}{6}(n-1)^4+\frac{1}{6}(n-1)^2$ squares in this domain.

In the same way, we can count to the end and record the results.

$$\frac{5}{6}n^4+\frac{1}{6}n^2,\quad \frac{5}{6}(n-1)^4+\frac{1}{6}(n-1)^2,\quad \ldots,\quad \frac{5}{6}1^4+\frac{1}{6}1^2.$$

The sum of all these values is equal to the total number of squares inside the rectangle. Namely

$$\frac{5}{6}(1^4+2^4+3^4+\ldots+n^4)+\frac{1}{6}(1^2+2^2+3^2+\ldots+n^2)=\frac{n(n+1)}{2}\cdot\frac{n(n+1)(2n+1)}{6}$$

$$\frac{5}{6}S_n^4=\frac{n\left(n+1\right)\left(2n+1\right)}{6}\left(\frac{n\left(n+1\right)}{2}-\frac{1}{6}\right)$$

$$S_n^4=\frac{n\left(n+1\right)\left(2n+1\right)(3n^2+3n-1)}{30}$$

Hence, $S_n^4=\sum_{k=1}^{n}k^4=\frac{n\left(n+1\right)\left(2n+1\right)(3n^2+3n-1)}{30}$.

Example 8. Calculate the sum $S_n^5=\sum_{k=1}^{n}k^5=1^5+2^5+3^5+\ldots+n^5$.

To solve the example, we consider a square shape whose width and height are equal, that is, both have $(1^2+2^2+3^2+\ldots+n^2)$ rows and columns. (Figure 7)

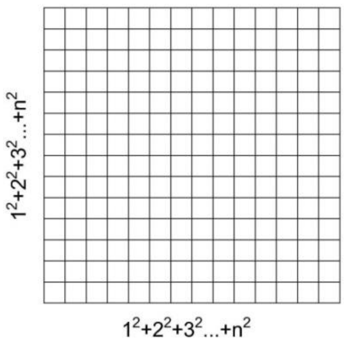


Figure 7

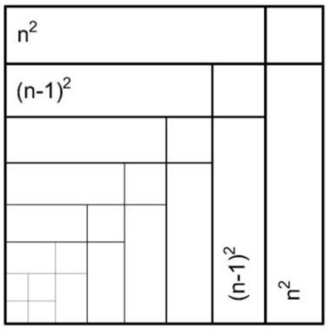


Figure 8

The total number of squares inside the square is

$$(1^2+2^2+3^2+\ldots+n^2)^2=\frac{n^2(n+1)^2(2n+1)^2}{36} \tag{7}$$

Now, we count the number of squares on each side of the square by separating n^2 columns and rows. (Figure 8)

$$2n^2(1^2+2^2+3^2+\ldots+n^2)-n^4=2n^2\frac{n\left(n+1\right)\left(2n+1\right)}{6}-n^4=\frac{2}{3}n^5+\frac{1}{3}n^3$$

By separating $(n-1)^2$ more such rows and columns from the rest of the square (Figure 8), we calculate the number of squares in it.

$$2(n-1)^2(1^2+2^2+3^2+...+(n-1)^2)-(n-1)^4=$$
$$=2(n-1)^2\frac{n(n-1)(2n-1)}{6}-(n-1)^4=\frac{2}{3}(n-1)^5+\frac{1}{3}(n-1)^3$$

Continuing this work to the end, we record the results obtained.

$$\frac{2}{3}n^5+\frac{1}{3}n^3,\quad \frac{2}{3}(n-1)^5+\frac{1}{3}(n-1)^3,\quad ...,\quad \frac{2}{3}1^5+\frac{1}{3}1^3.$$

Since these values represent the number of squares in the fields we have separated at each step, their sum is equal to the value given in (7) for the total number of squares inside the square in Figure 7.

$$\frac{2}{3}(1^5+2^5+3^5+...+n^5)+\frac{1}{3}\left(1^3+2^3+3^3+...+n^3\right)=\frac{n^2(n+1)^2(2n+1)^2}{36}$$
$$\frac{2}{3}S_n^5+\frac{1}{3}S_n^3=\frac{n^2(n+1)^2(2n+1)^2}{36}$$
$$2S_n^5=\frac{n^2(n+1)^2(2n+1)^2}{12}-\frac{n^2(n+1)^2}{4}$$
$$S_n^5=\frac{n^2(n+1)^2(2n^2+2n-1)}{12}$$

Hence, $S_n^5=\sum_{k=1}^{n}k^5=1^5+2^5+3^5+...+n^5=\frac{n^2(n+1)^2(2n^2+2n-1)}{12}$.

It can be concluded from the above examples. You can calculate any sum S_n^m $(m=1,2,3,\ldots)$ using the rectangular method. To do this, if $m=2r$ $(r\in N)$, it is enough to take the height $1^r+2^r+3^r+...+n^r$ and $1^{r-1}+2^{r-1}+3^{r-1}+...+n^{r-1}$ width of a rectangle (Figure 9).

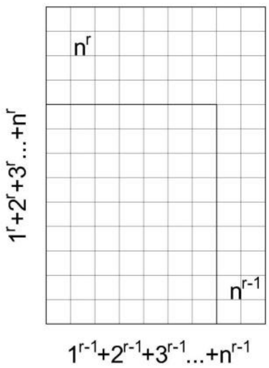


Figure 9

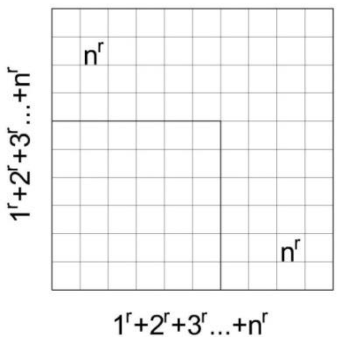


Figure 10

If $m=2r+1$ $(r=0,1,2,3,...)$, it is sufficient to obtain a rectangle in the form of a square with $1^r+2^r+3^r+...+n^r$ sides on both sides (Figure 10). The calculation process is similar to the examples above.

For example, it is easy to prove that the sum of $S_n^2=\sum_{k=1}^{n}k^2=1^2+2^2+3^2+...+n^2$ is equal to $\frac{n(n+1)(2n+1)}{6}$.

CONCLUSION

The introductory part of the article gives a brief account of the great Iranian mathematician al-Karaji, his contribution to the science of algebra, and the opinions left about him by historians.

The research methodology section discusses the areas of algebra studied by al-Karaji, his achievements, experiences, and innovations in this area. It contains the specific methods and results he used in the calculation of some sets, the characteristics of these sets, and evidence of the role of these works in the history of mathematics.

The problem-solving part of the article deals with the general methods of solving the sums formed by any degree of the sequence of natural numbers. First, the method of calculation using the well-known method, the formula of short multiplication, is described. Using this method, several sums were calculated and the general result was given.

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