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Approximate solution of the galerkin method for one non-classical problem of parabolic type

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Abstract: The article considers one boundary value problem of parabolic type with a divergent main part, when the boundary condition contains the time derivative of the desired function. Such non-classical problems arise in a number of applied problems, for example, when a homogeneous isotropic body is placed in the inductor of an induction furnace and an electromagnetic wave falls on its surface. Such problems have been little studied, so the study of problems of parabolic type, when the boundary condition contains the time derivative of the desired function, is relevant. The work defines a generalized solution to the problem under consideration in the space $\widetilde{\mathrm{H}^{1,1}}(Q_T)$. The purpose of the study is to prove the theorem of the existence and uniqueness of an approximate solution of the Bubnov-Galerkin method for the considered non-classical parabolic problem with a divergent main part, when the boundary condition contains the time derivative of the desired function.

Keywords: Mixed problems, quasilinear equation, boundary condition, Galerkin method, generalized solution, parabolic type, approximate solution, error estimate, a priori estimates, coordinate system, monotonicity, inequalities, boundary, domain, scalar product.

Introduction: When studying a number of current technical problems, it becomes necessary to study mixed parabolic problems, when the boundary condition contains a time derivative of the desired function. Problems of this type arise, for example, when a homogeneous isotropic body is placed in the inductor of an induction furnace and an electromagnetic wave falls on its surface. Some nonlinear problems of parabolic type with a boundary condition containing the time derivative of the desired function were considered, for example, in works [1-3]. Many scientists have been involved in constructing an approximate solution using the Galerkin method and obtaining a priori estimates of the approximate solution for parabolic classical quasilinear problems without a time derivative in the boundary condition: Mikhlin S.G., Douglas J. Jr., Dupont T., Dench J. E., Jr., Jutchell L., and others [4-7]. And quasilinear problems, when the boundary condition contains the time derivative of the desired function using the Galerkin method, are studied in works [8-12].

Statement of the problem. In this paper, we consider a quasilinear problem of parabolic type, when the boundary condition contains the time derivative of the desired function:

$$\begin{cases} u_t - \frac{d}{dx_i} a_i(x,t,u,\nabla u) + a(x,t,u,\nabla u) = 0 \ , \\ a_0 u_t + a_i(x,t,u,\nabla u)\cos(\nu, x_i) = g(x,t,u), \quad (x,t) \in S_t \ , \\ u(x,0) = u_0(x) \ , \ x \in \Omega \end{cases} \tag{1}$$

where $\Omega -$ *bounded domain in* E_2, $a_0 = const > 0, Q_T = \Omega \times [0,T], S_T = S \times [0,T]$, $S = \partial\Omega$

Definition. A generalized solution from the space $\widetilde{W_2^{1,1}}(Q_T) = \{U \in W_2^{1,1}(Q_T): a_0 U_t \in L_2(S_T)\}$ of problem (1) is a function from $\widetilde{W_2^{1,1}}(Q_T)$, satisfying the following identity

$$\int_{Q_T} (u_t\eta + a_i(x,t,u,\nabla u)\eta_{xi} + a(x,t,u,\nabla u)\eta)\, dxdt + \int_{S_T} (a_0 u_t + \quad + g(x,t,u)))\eta)\, dxdt = 0$$

(2)

$$\forall\, \eta \in W_2^{1}(\Omega)$$

Let us assume that the following conditions are satisfied:

A. *at* $(x,t,u,p) \in \{\bar{\Omega} \times [O,T] \times E_1 \times E_2\}$ functions $a_i(x,t,u,p), a(x,t,u,p)$ are measurable in (x,t,u,p), continuous in (t,u,p) and satisfy the inequalities

$$|a_i(x,t,u,p)| \le C\big(|P| + |U|^k\big) + \varphi_1(x,t)\,, \varphi_1 \in L_2(Q_T)\,, \quad i = 1,2 \tag{2.1}$$

$$|a(x,t,u,p)| \le C\big(|P|^{2-\epsilon} + |U|^k\big) + \varphi_2(x,t)\,, \quad \varphi_2 \in L_q(Q_T), \tag{3}$$

where $|P| = (\sum_{i=1}^{m} p_i^2)^{\frac{1}{2}}$, $k < \infty, \varepsilon > 0,\ q > 1$

B. The functions $a_i(x,t,u,p)$ have the form:

$$a_i(x,t,u,p) = \bar{a}_i(x,t,u,p) + \bar{\bar{a}}_i(x,p) \tag{4}$$

here

$$\bar{a}_i(x,t,u,p) = \frac{\partial \bar{a}(x,t,u,p)}{\partial p_i},$$

$$\left|\frac{\partial \bar{a}}{\partial t}\right| \le C(|u|^{2r} + |p|^2) + \varphi_3(x,t)\,, \quad \varphi_3 \in L_1(Q_T)$$

$$\left|\frac{\partial \bar{a}}{\partial u}\right| \le C(|u|^{r} + |p|) + \varphi_4(x,t)\,, \quad \varphi_4 \in L_2(Q_T) \tag{5}$$

$$r \ge 0 \quad , \quad \int_\Omega \bar{a}\,(x,t,u,\nabla u) dx \,\Big|_0^t \ge 0$$

C. For any smooth function $U(x,t)$ the inequality holds.

$$\int_{Q_T} \bar{\bar{a}}_i\,(x,\nabla U) U_{tx_i} \mathrm{dxdt} \ge \nu \|\nabla U\|^2_{L_2(\Omega)} \tag{6}$$

where ν- positive constant.

D. Monotonicity condition. For any functions u, $v \in W_2^{1}(\Omega)$

$$(a_i(x,t,u,\nabla u) - a_i(x,t,v,\nabla v), u_{x_i} - v_{x_i})_\Omega + + (a(x,t,u,\nabla u) - a(x,t,v,\nabla v), u - v)_\Omega \ge 0 \tag{7}$$

E. At $(x,t,u) \in \{\bar{\Omega} \times [o,T] \times E_1\}$ *function* $g(x,t,u)$ (t,u) is continuous in (t,u) and satisfies the inequality:

$$|g(x,t,u) - g(x,t,v)| \le g_0|u - v|, \qquad g(x,t,0) \in L_2(S_T) \tag{8}$$

Main results. Let us construct an approximate solution according to Galerkin [13-17]. Let's take a coordinate system from the space $W_2^{1}(\Omega)$. We will seek an approximate solution $U(x,t)$ in the form

$$U(x,t) = \sum_{k=1}^{n} C_k^n\,(t)\varphi_k(x) \tag{9}$$

where $C_k^n(t)$ are determined from the system of ordinary differential equations

$$(U_t, \varphi_j)_{\hat{L}_2} + (a_i(x,t,U,\nabla U), \varphi_{jx_i})_\Omega + (a(x,t,U,\nabla U), \varphi_j)_\Omega = $$
$$= (g(x,t,U), \varphi_j)_S \ , \quad j = \overline{1,n} \qquad (10)$$

and initial conditions

$$(U(x,O) - u_0, \varphi_j)_{W_2^1(\Omega)} = 0$$

Here $\hat{L}_2(\Omega)$ – space of functions with scalar product

$(u,v)_{\hat{L}_2} = (u,v)_\Omega + (u,v)_s \ , (u,v)_K = \int_K uvdx$

If the system $\{\varphi_k\}$ is orthonormal in the metric $\hat{L}_2(\Omega)$, then system (10) takes the form

$$\dot{C}_i^n = f_j^n(t, C_1^n, .., C_n^n), \qquad (11)$$

where $f_j^n(t, C_1^2, .., C_n^n) = -(a_i(x,t,U,\nabla U), \varphi_{jx_i})_\Omega - (a(x,t,U,\nabla U), \varphi_j)_\Omega + (g(x,t,U), \varphi_j)_S$

Theoreme.. If conditions A-E are satisfied, then there is a unique generalized solution to problem (1) in the space $\widetilde{W_2}^{1,1}(Q_T)$.

Proof. Condition A ensures the existence and continuity of the function $f_j^n(t, C_1^n, .., C_n^n)$ with respect to t and C_k^n.

Therefore, for the existence of at least one solution to problem (11) on the entire interval $[O,T]$, it is sufficient to know that all possible solutions are uniformly bounded. This limitation follows from the a priori assessment

$$\max_{0\le t\le T} \|U(x,t)\|^2_{\hat{L}_2} + \|U_t(x,t)\|^2_{L_{2(o,T,\hat{L}_2)}} + \max_{0\le t\le T} \|\nabla U(x,t)\|^2_{\hat{L}_2} \le N \qquad (12)$$

where is a constant that does not depend on n.

From here we obtain the inequality [18-19]

$$\max_{0\le t\le T} \|C_n(t)\|^2 = \max_{0\le t\le T} \|U(x,t)\|^2_{L_{2(\Omega)}} \le N, \quad C_n = \{C_k^n(t)\}_{k=1}^n$$

Let us now proceed to the limit transition with respect to $n \to \infty$. From estimate (12) it follows that there exists a function $u(x,t) \in \overline{W_2}^{1,1}(Q_T)$ and a subsequence $U(x,t)$, such that the functions $U(x,t)$converge to u(x,t) weakly in the norm $\overline{W_2}^{1,1}(Q_T)$ and the functions U_t converge to u_t in $L_2(S_t)$. Since the embeddings $\overline{W_2}^{1,1}(Q_T) \in L_2(Q_t), L_2(S_t)$ are compact, then $U(x,t) \to u(x,t)$ strongly in $L_2(S_t)$ and in $L_2(Q_t)$. From this convergence it follows that $U(x,t)$ converges to $u(x,t)$ in $L_2(\Omega)$ and in $L_2(S)$ for almost all t in $[O,T]$ and almost everywhere in $Q_t U S_t$.

Further, from condition A it follows that the functions $a_i(x,t,U,\nabla U)$ $i = 1,2$ converge weakly in $L_2(Q_T)$ and the elements $A_i(x,t)$ of the space $L_2(Q_T)$ and the functions $a(x,t,U,\nabla U)$ converge weakly $A(x,t) \in L_1(Q_T)$ in the space $L_1(Q_T)$.

Let us denote by P_l the set of linear combinations of the form

$$V(x,t) = \sum_{k=1}^{l} d_k(t)\varphi_k(x)$$

where $d_k(t)$ – are arbitrary smooth functions on the interval $[O,T]$. Multiplying relations (10) by $d_k(t)$, summing over k from 1 to l and integrating from 0 to t, we find that for any function $V(x,t) \in P_e$ the equality

$$\int_0^t (U_t, V)_{\hat{L}_2} dt + \int_{Q_t} a_i(x,t,U,\nabla U)V_{x_i} + a(x,t,U,\nabla U)V]dxdt == \int_{S_t} g(x,t,U)\, Vdxdt \qquad (13)$$

holds true.

Let's move on to the limit in n→∞. As a result we get:

$$\int_0^{\mathrm{T}}(U_t,V)_{\hat{L}_2}dt+\int_{Q_T}\left[A_i(x,t)V_{x_i}+A(x,t)V\right]dxdt=\int_{S_T}g(x,t,u)\,Vdxdt \qquad (14)$$

Since $\bigcup_{l=1}^{\infty}P_e$ is dense in ${W_2}^{1,0}(Q_T)$, then by performing the closure over V in (14), we obtain that equality (14) is valid for any function $V\in {W_2}^{1,0}(Q_T)$.

From equality (14) we obtain that the function $U(x,t)$ is the desired generalized solution.
Let's prove the uniqueness of the solution.

Let $U_1(x,t),U_2(x,t)$ be two solutions to problem (9), then their difference U_1-U_2 satisfies the relation

$$\int\limits_{Q_t}\frac{\partial(U_1-U_2)}{\partial t}(U_1-U_2)dxdt+a_0\int\limits_{S_t}\frac{\partial(U_1-U_2)}{\partial t}(U_1-U_2)dxdt$$

$$+\int\limits_{Q_T}\{[a_i(x,t,U_1,\nabla\mathrm{U}_1)-a_i(x,t,U_2,\nabla\mathrm{U}_2)](U_1-U_2)_{x_i}$$

$$+[a(x,t,U_1,\nabla U_1)-a(x,t,U_2,\nabla U_2)](U_1-U_2)\}dxdt$$

$$=\int\limits_{S_t}[g(x,t,U_1)-g(x,t,U_2)]\,(U_1-U_2)dxdt$$

Using conditions (5) and (7), we obtain

$$\int\limits_{\Omega}(U_1-U_2)^2\,dx+a_0\int\limits_{S}\int\limits_{\Omega}(U_1-U_2)^2\,dx\le 2\,g_0\int\limits_{S_T}\int\limits_{\Omega}(U_1-U_2)^2\,dxdt$$

Therefore $U_1\equiv U_2$. Thus, the theorem is proved.

CONCLUSION

In this paper, a generalized solution to the problem under consideration is defined in the space $\widetilde{\mathrm{H}^{1,1}}(Q_T)$ when the dimension of the domain in spatial variables is equal to two. Further, the existence and uniqueness theorem of an approximate solution of the Bubnov-Galerkin method for the considered non-classical parabolic problem with a divergent principal part is proved, when the boundary condition contains a time derivative of the desired function.

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